Symmetric irreducible representations of $SU(3)_q$ in the $SU(2)_q*U(1)$ basis and some extensions to $SU(N)_q \times SU(N-1)_q*U(1)$

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Symmetric irreducible representations of $SU(3)_q$ in the $SU(2)_q \times U(1)$ basis and some extensions to $SU(N)_q \supset SU(N-1)_q \times U(1)$

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Abstract. Irreducible tensor operators of the quantum group $SU(N)_q$ are defined by using the coproduct definition. The symmetric irreducible representations of $SU(3)_q$ in $SU(2)_q \times U(1)$ basis are discussed through the Jordan-Schwinger realizations. The extended Wigner-Eckart theorem is established, and a special class of isoscalar factors for $SU(3)_q \supset SU(2)_q \times U(1)$ is obtained, which can also be extended to the general $SU(N)_q \supset SU(N-1)_q \times U(1)$ case.

1. Introduction

Quantum groups [1-3] have recently been found to be important in many branches of physics [4-6]. Many new realizations of the quantum group $SU(N)_q$, especially $SU(2)_q$, have been obtained [7-14]. These results show that the properties of quantum groups are quite similar to those of classical Lie groups in the case of q not being a root of unity. However, it is not clear to what extent the familiar techniques used in the representation theory of classical groups are applicable to quantum groups: for example, the concept of irreducible tensor operators, and the generalized Wigner-Eckart theorem, which has been proved to be a powerful tool in the representation theory of classical groups. Very recently, the irreducible tensor operators for $SU(2)_q$ have been defined based on the coproduct rule of $SU(2)_q$ generators [15, 16]. In [15] the general definition of irreducible tensor operators is given in a somewhat abstract form, and the q-boson realizations of $SU(2)_q$ tensor operators are constructed only for spinor irreps. In [16] the irreducible tensor operators for $SU(2)_q$ are constructed by using q-boson operators, and the explicit definition of irreducible tensor operators and its reduced matrix elements are also presented.

On the other hand, it has been shown that rotational spectra of nuclei and molecules can be described very accurately in terms of a Hamiltonian which is proportional to the Casimir operator of the quantum group $SU(2)_q$ [17, 18]. As pointed out in [18], the quantum version of the SU(3) limit of the U(4) model for rotation-vibration spectra of diatomic molecules [19] can also be of interest.

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In this paper, we will consider symmetric irreducible representations of $SU(3)_q$ in $SU(2)_q \times U(1)$ basis as the starting point for studying the general $SU(N)_q$ in $SU(N-1)_q \times U(1)$ basis. The paper is organized as follows. In section 2, we discuss the quantum algebra $SU(3)_q$. Emphasis is on the realization of generators. In section 3, we give the definition of irreducible tensor operators for $SU(3)_q$ by using the coproduct rule. In section 4, we study the symmetric irreducible representations of $SU(3)_q$ in $SU(2)_q \times U(1)$ basis based on the Jordan–Schwinger realizations. We also establish an extended Wigner–Eckart theorem, and calculate isoscalar factors for $SU(3)_q \supset SU(2)_q \times U(1)$ for the coupling $\{n_1\} \times \{n_2\} \to \{n\}$. Finally, the results will be extended to the general $SU(N)_q$ case.

2. The quantum algebra $SU(3)_q$

The quantum algebra $SU(3)_q$ is achieved by defining generators for each co-root and simple root, the latter satisfying a q-analogue Serre relation [3]. A suitable basis of $SU(3)_q$ is the q-analogue of the Chevalley basis of SU(3). The generators are H_i $(i=1,2)$, and $X_i^\pm$ $(i=1,2,3)$, i.e. H_i are co-roots and $X_i^\pm$ are the generators corresponding to the roots α_i $(i=1,2,3)$ with inner product structure $(\alpha_i, \alpha_i)=2$, $(i=1,2)$, and $(\alpha_1, \alpha_2)=-1$.

The commutation relations are

$$
\begin{aligned}
&[H_i, X_i^\pm] = \pm 2X_i^\pm \qquad [H_1, X_2^\pm] = \mp X_2 \qquad [H_2, X_1^\pm] = \mp X_1^\pm \\
&[X_i^+, X_j^-] = \delta_{ij}[H_i] = \delta_{ij}\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} \qquad \text{for } i, j = 1, 2, 3.
\end{aligned} \tag{1}
$$

$X_i^\pm$ satisfy q-analogue Serre relations. However, we can define extra generators $X_3^\pm$ corresponding to the third root α_3 such that a direct use of this triple relation is avoided. The generators $X_3^\pm$ can be defined by a q-analogue of the adjoint action [20]

$$
X_3^\pm = \mathrm{ad}_q\, X_1^\pm X_2^\pm := q^{1/2} X_1^\pm X_2^\pm - q^{-1/2} X_2^\pm X_1^\pm. \tag{2}
$$

Then impose commutation relations between $X_3^\pm$ and $X_i^\pm$ $(i=1,2)$ such that the q-analogue Serre relations can be reproduced. The algebra structure of $X_3^\pm$ is given by

$$
[H_i, X_3^\pm] = \pm X_3^\pm \qquad i=1,2 \qquad [H_1 + H_2, X_3^\pm] = \pm 2X_3 \tag{3a}
$$

$$
\mathrm{ad}_q\, X_1^\pm X_3^\pm := q^{-1/2} X_1^\pm X_3^\pm - q^{1/2} X_3^\pm X_1^\pm = 0 \tag{3b}
$$

$$
\mathrm{ad}_q\, X_2^\pm X_3^\pm := q^{1/2} X_2^\pm X_3^\pm - q^{-1/2} X_3^\pm X_2^\pm = 0. \tag{3c}
$$

This definition of the adjoint action is taken from [20].

From [7, 12] one knows that the algebra $SU(3)_q$ can be realized by using the following algebra homomorphism:

$$
\begin{aligned}
&\tilde{E}_{11} - \tilde{E}_{22} \to H_1 \qquad \tilde{E}_{22} - \tilde{E}_{33} \to H_2 \\
&\tilde{E}_{12} \to X_1^+ \qquad \tilde{E}_{23} \to X_2^+ \qquad \tilde{E}_{13} \to X_3^+ \\
&\tilde{E}_{21} \to X_1^- \qquad \tilde{E}_{32} \to X_2^- \qquad \tilde{E}_{31} \to X_3^-
\end{aligned}
$$

where $\{\tilde{E}_{ij}\}$ satisfy the following commutation relations

$$
\begin{aligned}
&[\tilde{E}_{ij}, \tilde{E}_{ii}] = -\tilde{E}_{ij} \qquad [\tilde{E}_{ij}, \tilde{E}_{jj}] = \tilde{E}_{ij} \\
&[\tilde{E}_{ij}, \tilde{E}_{ji}] = [\tilde{E}_{ii} - \tilde{E}_{jj}] \qquad [\tilde{E}_{ii}, \tilde{E}_{jj}] = 0
\end{aligned} \tag{4a}
$$

for $i \neq j$. Especially,

$$[\tilde{E}_{13}, \tilde{E}_{23}] = [\tilde{E}_{32}, \tilde{E}_{31}] = [\tilde{E}_{12}, \tilde{E}_{13}] = [\tilde{E}_{21}, \tilde{E}_{23}] = 0$$
$$\tilde{E}_{12}\tilde{E}_{23} - \tilde{E}_{23}\tilde{E}_{12}q^{-1} = \tilde{E}_{13}q^{\tilde{E}_{22}} \tag{4b}$$
$$\tilde{E}_{13}\tilde{E}_{32} - \tilde{E}_{32}\tilde{E}_{13}q^{-1} = \tilde{E}_{12}q^{\tilde{E}_{33}}$$

etc, which can be realized by using q-deformed boson operators b_i^+, b_i, and N_i $(i = 1, 2, 3)$ with $\tilde{E}_{ij} = b_i^+ b_j$ [12]. The algebra generators $\{\tilde{E}_{ij}\}$ indeed satisfy the q-analogue Serre relations. But we do not know whether the generators $\{X_i^\pm\}$ and $\{\tilde{E}_{ij}\}$ are identical. This problem becomes crucial when one is concerned with the co-algebra structure of $SU(3)_q$.

The co-algebra structure is given by [20]

$$\Delta H_i = H_i \otimes 1 + 1 \otimes H_i \tag{5a}$$
$$\Delta X_i = X_i^\pm \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_i \qquad \text{for } i = 1, 2 \tag{5b}$$
$$\Delta X_3 = X_3^\pm \otimes q^{H_3/2} + q^{-H_3/2} \otimes X_3^\pm + (q - q^{-1})q^{-H_1/2} X_2^\pm \otimes q^{H_2/2} X_1^\pm \tag{5c}$$

where $X_i^\pm$ $(i = 1, 2, 3)$ satisfy the adjoint relations given by equations (2) and (3). It is obvious that $\{\tilde{E}_{ij}\}$ do not satisfy the adjoint relations. One can conclude that $\{\tilde{E}_{ij}\}$ and $\{X_i^\pm\}$ are not identical.

Using (4) and through direct calculation, we find the following realizations for $SU(3)_q$:

$$\begin{aligned} &H_1 = \tilde{E}_{11} - \tilde{E}_{22} && H_2 = \tilde{E}_{22} - \tilde{E}_{33} \\ &X_1^+ = \tilde{E}_{12},\ X_1^- = \tilde{E}_{21} && \\ &X_2^+ = \tilde{E}_{23} q^{1/2(\tilde{E}_{11} + \tilde{E}_{22})} && X_2^- = q^{-1/2(\tilde{E}_{11} + \tilde{E}_{22})} \tilde{E}_{32} \\ &X_3^+ = \tilde{E}_{13} q^{1/2(\tilde{E}_{11} + 3\tilde{E}_{22} + 1)} && X_3^- = q^{-1/2(\tilde{E}_{11} + 3\tilde{E}_{22} + 1)} \tilde{E}_{31}. \end{aligned} \tag{6}$$

One can easily verify that this realization can reproduce the adjoint relations given by (2) and (3). We will use this realization in this paper.

As in the $SU(3) \supset SU(2) \times U(1)$ case, we choose $SU(2)_q \times U(1)$ with generators $\{H_1, X_1^\pm, \hat{Y} = (H_2 + H_3)/3\}$ to be the subgroup of $SU(3)_q$. The symmetric irreducible representations of $SU(3)_q$ in the $SU(2)_q \times U(1)$ basis will be discussed in section 4.

3. Irreducible tensor operators

We assume that $\{T_{jmY}^{\{n_1 n_2\}}(q)\}$ is a set of irreducible tensor operators for $SU(3)_q$, which spans the irreducible representation $\{n_1 n_2\}$ of $SU(3)_q$, where j is the angular momentum quantum number for $SU(2)_q$, and m is its U(1) label, and Y is the quantum number of supercharge. Basis vectors for $SU(3)_q \supset SU(2)_q \times U(1)$ can be written as

$$\left| \begin{matrix} \{n_1' n_2'\} \\ j' m' Y' \end{matrix} \right\rangle_q . \tag{7}$$

Then, from the coproduct definition given by (5) we have

$$X_1^\pm T_{jmY}^{\{n_1 n_2\}}(q) \left| \begin{matrix} \{n_1' n_2'\} \\ j' m' Y' \end{matrix} \right\rangle_q$$
$$= (X_1^\pm T_{jmY}^{\{n_1 n_2\}}(q)) q^{\frac{1}{2}H_1} \left| \begin{matrix} \{n_1' n_2'\} \\ j' m' Y' \end{matrix} \right\rangle_q$$
$$+ T_{jmY}^{\{n_1 n_2\}}(q) X_1^\pm q^{-m} \left| \begin{matrix} \{n_1' n_2'\} \\ j' m' Y' \end{matrix} \right\rangle_q \tag{8a}$$

$$X_2^\pm T_{jmY}^{\{n_1n_2\}}(q)\left|\begin{matrix}\{n_1'n_2'\}\\ j'm'Y'\end{matrix}\right\rangle_q$$
$$=(X_2^\pm T_{jmY}^{\{n_1n_2\}}(q))q^{\frac{1}{2}H_2}\left|\begin{matrix}\{n_1'n_2'\}\\ j'm'Y'\end{matrix}\right\rangle_q$$
$$+T_{jmY}^{\{n_1n_2\}}(q)X_2^\pm q^{\frac{1}{2}m-3Y/4}\left|\begin{matrix}\{n_1'n_2'\}\\ j'm'Y'\end{matrix}\right\rangle_q \tag{8b}$$

and

$$H_iT_{jmY}^{\{n_1n_2\}}(q)\left|\begin{matrix}\{n_1'n_2'\}\\ j'm'y'\end{matrix}\right\rangle_q$$
$$=(H_iT_{jmY}^{\{n_1n_2\}}(q))\left|\begin{matrix}\{n_1'n_2'\}\\ j'm'Y'\end{matrix}\right\rangle_q+T_{jmY}^{\{n_1n_2\}}(q)H_i\left|\begin{matrix}\{n_1'n_2'\}\\ j'm'Y'\end{matrix}\right\rangle_q \tag{8c}$$

from which we obtain

$$(X_1^\pm T_{jmY}^{\{n_1n_2\}}(q))=(X_1^\pm T_{jmY}^{\{n_1n_2\}}(q)-T_{jmY}^{\{n_1n_2\}}(q)X_1^\pm q^{-m})q^{-\frac{1}{2}H_1} \tag{9a}$$
$$(X_2^\pm T_{jmY}^{\{n_1n_2\}}(q))=(X_2^\pm T_{jmY}^{\{n_1m_2\}}(q)-T_{jmY}^{\{n_1n_2\}}(q)X_2^\pm q^{\frac{1}{2}m-3Y/4})q^{-\frac{1}{2}H_2} \tag{9b}$$
$$(H_iT_{jmY}^{\{n_1n_2\}}(q))=[H_i,\ T_{jmY}^{\{n_1n_2\}}(q)] \tag{9c}$$

which can be taken as the definition of irreducible tensor operators for $\mathrm{SU}(3)_q$. Equation (9) uniquely defines the irreducible tensor operator $T_{jmY}^{\{n_1n_2\}}(q)$ because the generators $X_3^\pm$ are defined by the adjoint action (2). Using (5), we similarly obtain

$$(X_3^\pm T_{jmY}^{\{n_1n_2\}}(q))$$
$$=(X_3^\pm T_{jmY}^{\{n_1n_2\}}(q)-T_{jmY}^{\{n_1n_2\}}(q)X_3^\pm q^{-\frac{1}{2}m-3Y/4})q^{-\frac{1}{2}H_3}+(q-q^{-1})q^{-m}$$
$$\times(X_2^\pm T_{jmY}^{\{n_1n_2\}}(q)-T_{jmY}^{\{n_1n_2\}}(q)X_2^\pm q^{\frac{1}{2}m-3Y/4})X_1^\mp q^{-\frac{1}{2}H_3}. \tag{10}$$

Using the results given by [16], we have

$$\begin{aligned}(X_1^\pm T_{jmY}^{\{n_1n_2\}}(q))&=\{[j\mp m][j\pm m+1]\}^{1/2}T_{jm\pm 1Y}^{\{n_1n_2\}}(q)\\ (H_1T_{jmY}^{\{n_1n_2\}}(q))&=2mT_{jmY}^{\{n_1n_2\}}(q)\\ (\hat{Y}T_{jmY}^{\{n_1n_2\}}(q))&=YT_{jmY}^{\{n_1n_2\}}(q).\end{aligned} \tag{11}$$

However, $(X_2^\pm T_{jmY}^{\{n_1n_2\}}(q))$ needs to be determined. Results for symmetric irreps of $\mathrm{SU}(3)_q$ will be given in the next section.

4. Symmetric irreducible representations of $\mathrm{SU}(3)_q$ in $\mathrm{SU}(2)_q\times\mathrm{U}(1)$ basis

Firstly, let us introduce an associate boson algebra $\mathscr{B}(q)$ with units generated by elements b_i, b_i^+, and N_i $(i=1,2,3)$, which satisfy the relations

$$\begin{aligned}&[N_i,\ b_j^+]=\delta_{ij}b_j^+ &&[N_i,\ b_j]=-\delta_{ij}b_j\\ &[b_i,\ b_j]=[b_j^+,\ b_i^+]=0 &&b_ib_i^+-q^{\pm1}b_i^+b_i=q^{\mp N_i}.\end{aligned} \tag{12}$$

In the following, we suppose that q is real. The star operation in $\mathscr{B}(q)$ is

$$(b_i)^\dagger=b_i^+\qquad (b_i^+)^\dagger=b_i\qquad N_i^\dagger=N_i. \tag{13}$$

Thus the relations given by (12) are invariant under the star anti-involution. Other useful relations are

$$q^{\pm N_i} b_i^+ q^{\mp N_i} = b_i^+ q^{\pm 1} \qquad q^{\pm N_i} b_i q^{\mp N_i} = b_i q^{\mp 1} \tag{14a}$$

and

$$b_i b_i^{+k} = b_i^{+k} b_i q^{\pm k} + [k] b_i^{+k-1} q^{\mp N_i} \qquad b_i^{+k} b_i = b_i^{+k-1}[N_i] \tag{14b}$$

which can be derived using (12).

Using these operators we obtain the following Jordan–Schwinger realizations for $SU(3)_q$:

$$\begin{aligned} &H_1 = N_1 - N_2 \qquad H_2 = N_2 - N_3 \\ &X_1^+ = b_1^+ b_2 \qquad X_1^- = b_2^+ b_1 \\ &X_2^+ = b_2^+ b_3 q^{\frac{1}{2}(N_1+N_2)} \qquad X_2^- = q^{-\frac{1}{2}(N_1+N_2)} b_3^+ b_2 \\ &X_3^+ = b_1^+ b_3 q^{\frac{1}{2}(N_1+3N_2+1)} \qquad X_3^- = q^{-\frac{1}{2}(N_1+3N_2+1)} b_3^+ b_1 . \end{aligned} \tag{15}$$

$SU(3)_q \supset SU(2)_q \times U(1)$ orthonormal basis vectors for the symmetric irrep $\{n\}$ can be written as

$$|njm\rangle_q \equiv \left| \begin{matrix} \{n0\} \\ jm;\ Y \end{matrix} \right\rangle_q = \frac{b_1^{+j+m} b_2^{+j-m} b_3^{+n-2j}}{\{[n-2j]![j+m]![j-m]!\}^{1/2}} |0\rangle_q \tag{16}$$

where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, n/2$, and the quantum number Y will be omitted because Y can be expressed in terms of quantum number j, namely

$$Y = 2j - 2n/3. \tag{17}$$

Using the definition for irreducible tensor opertors given by (9), we can prove that

$$T_{jm}^n(q) = \frac{b_1^{+j+m} b_2^{+j-m} b_3^{+n-2j}}{\{[n-2j]![j+m]![j-m]!\}^{\frac{1}{2}}} q^{\frac{1}{2}N_1(j-m) - \frac{1}{2}N_2(j+m) - N_3(j+\frac{1}{3}n)} \tag{18}$$

are symmetric tensor operators for $SU(3)_q$. Taking the procedure outlined in [16], we obtain

$$\begin{aligned} &[\tfrac{1}{2}H_1, T_{jm}^n(q)] = m T_{jm}^n(q) \\ &[H_2, T_{jm}^n(q)] = (3j - m - n) T_{jm}^n(q) \\ &(X_1^\pm T_{jm}^n(q) - T_{jm}^n(q) X_1^\pm q^{-m}) q^{-\frac{1}{2}H_1} = \{[j \mp m][j \pm m + 1]\}^{\frac{1}{2}} T_{jm\pm1}^n(q) \\ &(X_2^+ T_{jm}^n(q) - T_{jm}^n(q) X_2^+ q^{-\frac{1}{2}(3j-m-n)}) q^{-\frac{1}{2}H_2} \\ &\qquad = q^j \{[n-2j][j-m+1]\}^{\frac{1}{2}} T_{j+\frac{1}{2} m-\frac{1}{2}}^n(q) \\ &(X_2^- T_{jm}^n(q) - T_{jm}^n(q) X_2^- q^{-\frac{1}{2}(3j-m-n)}) q^{-\frac{1}{2}H_2} \\ &\qquad = q^{-j+\frac{1}{2}} \{[n-2j+1][j-m]\}^{\frac{1}{2}} T_{j-\frac{1}{2} m+\frac{1}{2}}^n(q) \end{aligned} \tag{19}$$

which gives the definition for symmetric irreducible tensor operators for $SU(3)_q$.

By using the Wigner–Eckart theorem for irreducible tensor operators for $SU(2)_q$ given by [16], matrix elements of $T_{j_2 m_2}^{n_2}(q)$ can be written as

$$_q\langle njm| T_{j_2 m_2}^{n_2}(q) |n_1 j_1 m_1\rangle_q = \langle nj \| T_{j_2}^{n_2} \| n_1 j_1\rangle_q \langle j_1 m_1 j_2 m_2 | jm\rangle_q \tag{20}$$

where $\langle j_1 m_1 j_2 m_2 | jm\rangle_q$ is the $SU(2)_q$ CG coefficient, and $\langle nj \| T^{n_2}_{j_2} \| n_1 j_1\rangle_q$ is the $SU(2)_q$ reduced matrix element. The matrix element given by (20) can also be directly calculated using (16) and (18), and can further be expressed as

$$
\begin{aligned}
&{}_q\langle njm | T^{n_2}_{j_2 m_2}(q) | n_1 j_1 m_1\rangle_q \\
&\quad = \langle n \| T^{n_2} \| n_1\rangle_q q^{\Delta(j_1 j_2 j)} \begin{bmatrix} n_1 & n_2 & n \\ j_1 & j_2 & j \end{bmatrix}_q \langle j_1 m_1 j_2 m_2 | jm\rangle_q
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
\begin{bmatrix} n_1 & n_2 & n \\ j_1 & j_2 & j \end{bmatrix}_q &= q^{-n_1 j_2 + n_2 j_1} \\
&\quad \times \left\{ \frac{[2j_1+2j_2]![n_1+n_2-2j_1-2j_2]![n_1]![n_2]!}{[n_1+n_2]![n_1-2j_1]![n_2-2j_2]![2j_1]![2j_2]!} \right]^{1/2}.
\end{aligned} \tag{22}
$$

Now

$$
\langle n \| T^{n_2} \| n_1\rangle_q = q^{-\frac{1}{2}n_1 n_2} \left\{ \frac{[n_1+n_2]!}{[n_1]![n_2]!} \right\}^{1/2} \tag{23}
$$

can be regarded as the $SU(3)_q$ reduced matrix element, and

$$
q^{\Delta(j_1 j_2 j)} = q^{2 j_1 j_2} \tag{24}
$$

which is determined for symmetric irresps only. Equation (21) gives an extended Wigner–Eckart theorem for the quantum group $SU(3)_q$ in the $SU(2)_q \times U(1)$ basis.

It can be verified that isoscalar factors for $SU(3)_q \supset SU(2)_q \times U(1)$ given by (22) indeed satisfy the orthogonality relation

$$
\sum_{j_1 j_2} \begin{bmatrix} n_1 & n_2 & n \\ j_1 & j_2 & j \end{bmatrix}_q^* \begin{bmatrix} n_1 & n_2 & n \\ j_1 & j_2 & j' \end{bmatrix}_q = \delta_{jj'}. \tag{25}
$$

Now we turn to matrix elements of $SU(3)_q$ generators $X_i^{\pm}$ $(i=2,3)$. It should be noted that X_i^+ and X_i^- with $i=2,3$ are not $SU(2)_q$ spinor operators, but we can construct the following $SU(2)_q$ spinor operators

$$
\begin{aligned}
T^{1/2}_{1/2}(q) &= X_3^+ q^{-\frac{1}{2}(\tilde{E}_{11}+4\tilde{E}_{22}+1)} = \tilde{E}_{13} q^{-\tilde{E}_{22}/2} \\
T^{1/2}_{-1/2}(q) &= X_2^+ q^{-\frac{1}{2}(2E_{11}+E_{22})} = \tilde{E}_{23} q^{\tilde{E}_{11}/2}
\end{aligned} \tag{26a}
$$

and

$$
\begin{aligned}
V^{1/2}_{1/2}(q) &= X_2^- q^{\frac{1}{2}(2\tilde{E}_{11}+\tilde{E}_{22}-1)} = \tilde{E}_{32} q^{\tilde{E}_{11}/2} \\
V^{1/2}_{-1/2}(q) &= -X_3^- q^{1/2(\tilde{E}_{11}+2\tilde{E}_{22}-2)} = -\tilde{E}_{31} q^{-\tilde{E}_{22}/2-1}.
\end{aligned} \tag{26b}
$$

They satisfy the relation [16]

$$
(V^{1/2}_m(q))^\dagger = (-1)^{\frac{1}{2}-m} q^{-(\frac{1}{2}-m)} T^{1/2}_{-m}(q). \tag{27}
$$

The matrix elements of $X_i^{\pm}$ $(i=2,3)$ can be written as

$$
\begin{aligned}
{}_q\langle nj'm' | X_2^+ | njm\rangle_q &= q^{\frac{1}{2}(3j-m)} \langle nj' \| T^{1/2} \| nj\rangle_q \delta_{j'j+\frac{1}{2}} \langle jm \tfrac{1}{2} -\tfrac{1}{2} | j'm'\rangle_q \\
{}_q\langle nj'm' | X_2^- | njm\rangle_q &= q^{-j+\frac{1}{2}-\frac{1}{2}(j+m)} \langle nj' \| V^{1/2} \| nj\rangle_q \delta_{j'j-\frac{1}{2}} \langle jm \tfrac{1}{2} \tfrac{1}{2} | j'm'\rangle_q \\
{}_q\langle nj'm' | X_3^+ | njm\rangle_q &= q^{j+\frac{1}{2}+3(j-m)/2} \langle nj' \| T^{1/2} \| nj\rangle_q \delta_{j'j+\frac{1}{2}} \langle jm \tfrac{1}{2} \tfrac{1}{2} | j'm'\rangle_q \\
{}_q\langle nj'm' | X_3^- | njm\rangle_q &= q^{-j+1-\frac{1}{2}(j-m)} \langle nj' \| V^{1/2} \| nj\rangle_q \delta_{j'j-\frac{1}{2}} \langle jm \tfrac{1}{2} -\tfrac{1}{2} | j'm'\rangle_q
\end{aligned} \tag{28}
$$

where

$$\langle nj+\tfrac{1}{2}\| T^{1/2} \| nj\rangle_q = \{[n-2j][2j+1]\}^{1/2}$$
$$\langle nj-\tfrac{1}{2}\| V^{1/2} \| nj\rangle_q = -\{q^{-1}[n-2j+1][2j+1]\}^{1/2} \tag{29}$$

are $SU(2)_q$ reduced matrix elements defined in [16].

5. Extension to $SU(N)_q \supset SU(N-1)_q \times U(1)$

In this paper, the $SU(3)_q$ irreducible tensor operators are defined, and a special class of symmetric irreducible tensor operators of $SU(3)_q$ in $SU(2)_q \times U(1)$ basis is realized based on the Jordan–Schwinger q-deformed boson realizations. We find that the tensor operators can be defined from the coproduct rule of $SU(3)_q$ generators. We also obtain an extended Wigner–Eckart theorem for the symmetric irreducible tensor operators. The new feature, now a q-factor $q^{\Delta(j_1j_2j)}$, comes into play. All these results contract to those in the $SU(3) \supset SU(2) \times U(1)$ case when $q \to 1$. We can infer that the isoscalar factors for $SU(N)_q \supset SU(N-1)_q$ for the coupling $\{n_1\dot{0}\} \times \{n_2\dot{0}\} \to \{n\dot{0}\}$ can be obtained from (22) with the analytical continuation $2j_i \to n_i'$ $(i=1,2)$, and $2j \to n'$, which can be expressed as

$$\left[\begin{array}{c|c|c} SU(N)_q & \{n_1\dot{0}\}\{n_2\dot{0}\} & \{n\dot{0}\} \\ SU(N-1)_q & \{n_1'\dot{0}\}\{n_2'\dot{0}\} & \{n'\dot{0}\} \end{array}\right]_q = q^{-\frac{1}{2}(n_1n_2'-n_2n_1')} \times \left\{\frac{[n_1'+n_2']![n_1+n_2-n_1'-n_2']![n_1]![n_2]!}{[n_1+n_2]![n_1-n_1']![n_1']![n_2-n_2']![n_2']!}\right\}^{1/2}. \tag{30}$$

Finally, we present a definition for $SU(N+1)_q$ irreducible tensor operators. The algebra $SU(N+1)_q$ is generated by $X_i^{\pm}$, H_i $(i=1,2,\ldots,n)$ under the relations

$$[H_i, H_j] = 0 \tag{31a}$$
$$[X_i^+, X_j^-] = \delta_{ij} H_i \tag{31b}$$
$$[X_i^+, X_j^+] = [X_i^-, X_j^-] = 0 \qquad \text{if } a_{ij} = 0 \tag{31c}$$
$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{31d}$$
$$X_i^{\pm 2} X_j^{\pm} - (q+q^{-1}) X_i^{\pm} X_j^{\pm} X_i^{\pm} + X_j^{\pm} X_i^{\pm 2} = 0 \qquad \text{if } a_{ij} = -1 \tag{31e}$$

where $a_{ij} = 2\delta_{ij} - \delta_{ij+1} - \delta_{ij-1}$ is an element of the Cartan matrix of $SU(N+1)$. The coproduct rule is

$$\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1$$
$$\Delta(X_i) = X_i^{\pm} \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_i^{\pm}. \tag{32}$$

Let $\{T_m^{\{p\}}(q)\}$ be a set of irreducible tensor operators of $SU(N+1)_q$, which spans an irreducible representation $\{p\}$ of $SU(N+1)_q$, where $\{p\} \equiv \{p_1p_2\ldots p_N\}$ and m is a set of quantum numbers needed in labelling the basis. For convenience, we assume

$$(H_i T_m^{\{p\}}(q)) = h_i T_m^{\{p\}}(q) \qquad \text{for } i = 1, 2, \ldots, N. \tag{33}$$

Then the irreducible tensor operators $T_m^{\{p\}}(q)$ satisfy the following definition

$$[H_i, T_m^{\{p\}}(q)] = h_i T_m^{\{p\}}(q) \tag{34a}$$
$$(X_i^{\pm} T_m^{\{p\}}(q) - T_m^{\{p\}}(q) X_i^{\pm} q^{-\frac{1}{2}h_i}) q^{-\frac{1}{2}H_i} = (X_i^{\pm} T_m^{\{p\}}(q)) \tag{34b}$$

for $i=1,2,\ldots,N$. As has been proved in [16], $(X_i^{\pm}T_m^{\{p\}}(q))$ can be obtained via $X_i^{\pm}|\{p\}m\rangle_q$. Thus to determine $(X_i^{\pm}T_m^{\{p\}}(q))$ needs detailed knowledge of $\mathrm{SU}(N+1)_q$ irreducible representations.

It should be noted that the generators $X_i^{\pm}$ in (32) should satisfy some additional conditions when they are used to define the irreducible tensor operators of $\mathrm{SU}(N+1)_q$. For example, in the $\mathrm{SU}(3)_q$ case the generators $\tilde{E}_{23}(\tilde{E}_{32})$ indeed satisfy the relations $(31a\text{-}e)$, but one cannot obtain a self-consistent definition for $\mathrm{SU}(3)_q$ irreducible tensor operators when one takes them as $X_2^+(X_2^-)$. In this case the additional conditions are the adjoint relations given by equation (2) and (3). Generators satisfying the q-analogue Serre relations may not satisfy the adjoint relations. However, if generators satisfy the adjoint relations (2) and (3), they can also reproduce the q-analogue Serre relations given by $(31e)$. That means a different choice of generators $X_i^{\pm}$ may be possible. $X_i^{\pm}$, $X_i'^{\pm},\ldots$, etc, may all satisfy the relations (31), but the algebra structure of each are quite different when they are endorsed with the coproduct. However, in order to give a self-consistent definition for irreducible tensor operators, the choice of generators $X_i^{\pm}$ which can be endorsed with the coproduct structure is unique. The additional relations for $X_i^{\pm}$ in the $\mathrm{SU}(N+1)_q$ case may be adjoint relations similar to those given in the $\mathrm{SU}(3)_q$ case.

Note added in proof. After completion of this work, we recived some preprints from Professor L C Biedenharn. In [21], the induced representations of $\mathrm{SU}(3)_q$ in the $\mathrm{SU}(2)_q\times \mathrm{U}(1)$ basis are constructed using the analogue of the Borel–Weil construction, but the tensor operators of $\mathrm{SU}(3)_q$ are not discussed. The authors are grateful to Professor Biedenharn for sending us these preprints.

References

[1] Kulish P P and Reshetikhin N Yu 1983 *J. Sov. Math.* **23** 2435
[2] Drinfeld V 1985 *Sov. Math. Dokl.* **32** 254
[3] Drinfeld V 1986 *Proc. Int. Congr. Math.* vol 1 (Berkeley, CA: University of California Press) pp 798–820
[4] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[5] Verlinde E 1988 *Nucl. Phys.* B **300** 360
[6] Knizhnik V and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
[7] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[8] Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613
[9] Takhtajan L A 1989 *Adv. Stud. Pure Math.* **19** 435
[10] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[11] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[12] Sun C-P and Fu H-C 1989 *J. Phys. A: Math. Gen.* **22** L983
[13] Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
[14] Pan F 1991 *Chin. Phys. Lett.* **8** 56
[15] Biedenharn L C and Tarlini M 1990 *Lett. Math. Phys.* **20** 271
[16] Pan F 1991 *J. Phys. A: Math. Gen.* **24** L803
[17] Bonatsos D, Raychev P P, Roussev R P and Smirnov Yu F 1991 *Chem. Phys. Lett.* **17** 300
[18] Raychev P P, Roussev R P and Smirnov Yu F 1990 *J. Phys. G: Nucl. Phys.* **16** L137
[19] Iachello F and Levine R D 1982 *J. Chem. Phys.* **77** 3046
[20] Burroughs N 1990 *Commun. Math. Phys.* **127** 109
[21] Biedenharn L C and Lohe M A 1991 *Preprint* Duke University